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On a Hypersurface of a Generalized (α, β)- Metric Space

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Abstract

In 1985, Matsumoto. M.,[6] has discussed the properties of special hypersurface of Rander's space with $b_i(x) = \partial_i\, b$ being the gradient of a scalar function $b(x)$. He had considered a hypersurface which is given by $b(x) =$ constant. In this paper we have considered the hypersurface of a generalized (α, β)-metric space with metric given by (1.1) which is given by the same equation $b(x) =$ constant. The condition under which this hypersurface be a hypersurface of the first, second and third kind have also been obtained.

Keywords: (α, β)- metric, hypersurface, angular metric, the reciprocal tensor, covariant differentiation, h- and v- covariant derivatives.

1. Introduction

The notion of (α, β)- metric was introduced in 1972 by Matsumoto. M., [5 & 6]. On the basis of Rander's metric which was attracted physicist's attention [3, 4]. A Finsler metric $L\,(x, y)$ in a differential manifold M^n is called (α, β)- metric, if the L is a (1) p-homogenous in the variables α and β,

where $\alpha = \sqrt{a_{ij}\,(x)\, y^i\, y^j}$ and $\beta = b_i\,(x)\, y^i$ is one form of degree one.

We have a number of (α, β)- metric such as Rander's metric, Kropina metric, generalized Kropina metric, Matsumoto metric as examples. With respect to these metrics, several authors ([4], [5], [6],[7], [10], [11], [12]) where we obtained important result and theorems. In this paper, we take the (α, β)-metric given by,

$$L^n = C_1\, \alpha^n + C_2\, \alpha^{n-1}\, \beta + C_3\, \alpha\, \beta^{n-1} + C_4\, \beta^n \tag{1.1}$$

where C_1, C_2, C_3 and C_4 are constants and n is a positive integer.

If $C_1 = C_4 = 1$, $C_2 = C_3 = 0$ and $n = 1$, then we get Rander's metric

$L = \alpha + \beta$ [11]

In this way by giving different values to $C_1, C_2, C_3\, C_4$ and n we get different type of (α, β)- metric discussed by several authors [8], [9] etc earlier. Therefore, the metric (1.1) has become too much interesting because it is the generalization of several (α, β)- metric. Therefore, we say this metric as generalized (α, β)- metric and space generalized (α, β)- metric.

In 1985, Matsumoto. M.,[6] has discussed the properties of special hypersurface of Rander's space with $b_i(x) = \partial_i\, b$ being the gradient of a scalar function $b(x)$. He had considered a hypersurface which is given by $b(x) =$ constant.

In this paper we have considered the hypersurface of a generalized (α,β)- metric space with metric given by (1.1) which is given by the same equation $b(x) =$ constant.

The conditions under which this hypersurface be a hypersurface of the first, second and third kinds have also been obtained.

2. Preliminaries.

Let $F^n = (M^n, L)$ be an n-dimensional Finsler spaces with (α,β) given by (1.1) where

$\alpha = \sqrt{a_{ij}\,(x)\, y^i\, y^j}$ is a Riemannian metric in M^n and $\beta = b_i\,(x)\, y^i$ is a differential one form in M^n.

The derivatives of $L = (\alpha,\beta)$ with respect to and are given by

$$L_\alpha = L^{1-n}\left[C_1\,\alpha^{n-1} + \frac{C_2\,(n-1)}{n}\alpha^{n-2}\,\beta + \frac{C_3}{n}\,\beta^{n-1}\right] \tag{2.1}$$

$$L_\beta = L^{1-n}\left[\frac{C_2\,\alpha^{n-1}}{n} + \frac{C_3\,(n-1)}{n}\beta^{n-2}\,\alpha + C_4\,\beta^{n-1}\right]$$

$$L_{\alpha\alpha} = L^{1-n}\left[C_1(n-1)\,\alpha^{n-2} + \frac{C_2\,(n-1)\,(n-2)\,\alpha^{n-3}\,\beta}{n}\right] + (1-n)L^{-1}L_\alpha^2 \tag{2.2}$$

$$L_{\beta\beta} = L^{1-n}\left[\frac{C_3\,(n-1)\,(n-2)\,\alpha\,\beta^{n-3}}{n} + C_4(n-1)\,\beta^{n-2}\right] + (1-n)\,L^{-1}\,L_\beta^2$$

$$L_{\alpha\beta} = L^{1-n}\left[\frac{C_2\,(n-1)\,\alpha^{n-2}}{n} + \frac{C_3\,(n-1)\,\beta^{n-2}}{n}\right] + (1-n)\,L^{-1}\,L_\alpha\,L_\beta$$

Where

$$L_\alpha = \frac{\partial L}{\partial \alpha}, \qquad L_\beta = \frac{\partial L}{\partial \beta}, \qquad L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha}$$

$$L_{\beta\beta} = \frac{\partial L_\beta}{\partial \beta} \qquad \text{and} \qquad L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}$$

The normalized element of support $l_i = \dot{\partial}_i\, L$ is given by

$$l_i = L_\alpha\, y_i\, \alpha^{-1} + L_\beta\, b_i$$

$$l_i = L^{1-n}\left[C_1\,\alpha^{n-1} + \frac{C_2\,(n-1)}{n}\alpha^{n-2}\,\beta + \frac{C_3}{n}\,\beta^{n-1}\right] y_i\,\alpha^{-1}$$

$$+ L^{1-n}\left[\frac{C_2\,\alpha^{n-1}}{n} + \frac{C_3\,(n-1)}{n}\beta^{n-2}\,\alpha + C_4\,\beta^{n-1}\right] b_i \tag{2.3}$$

Where $y_i = a_{ij}y^j$, the angular metric tensor $h_{ij} = L\,\dot{\partial}_i\,\dot{\partial}_j\,L$ is given by

$$h_{i_j} = P_0\,a_{ij} + q_0\,b_i\,b_j + q_{-1}\left(b_i\,y_j + b_j\,y_i\right) + P_{-2}\,y_i\,y_j \tag{2.4}$$

Where

$$P_0 = \frac{LL_\alpha}{\alpha} = \alpha^{-1}\,L^{2-n}\left[C_1\,\alpha^{n-1} + \frac{C_2\,(n-1)}{n}\alpha^{n-2}\,\beta + \frac{C_3}{n}\,\beta^{n-1}\right]$$

$$q_0 = LL_{\beta\beta} = L^{2-n}\left[\frac{C_3\,(n-1)\,(n-2)\,\alpha\,\beta^{n-3}}{n} + C_4(n-1)\,\beta^{n-2}\right] + (1-n)\,L_\beta^2$$

(2.5)

$$q_{-1} = \frac{LL_{\alpha\beta}}{\alpha} = \alpha^{-1}\left[L^{2-n}\left\{\frac{C_2\,(n-1)\,\alpha^{n-2}}{n} + \frac{C_3\,(n-1)\,\beta^{n-2}}{n}\right\} + (1-n)\,L_\alpha\,L_\beta\right]$$

$$P_{-2} = \frac{L}{\alpha^2}\left(L_{\alpha\alpha} - \frac{L_\alpha}{\alpha}\right) = \frac{L}{\alpha^2}\left[\begin{array}{c} L^{1-n}\left\{\left(C_1(n-1)\,\alpha^{n-2} + \frac{C_2\,(n-1)\,(n-2)\,\alpha^{n-3}\,\beta}{n}\right)\right\} + (1-n)L^{-1}L_\alpha^2 \\ -\frac{L^{1-n}\left(C_1\,\alpha^{n-1} + \frac{C_2\,(n-1)}{n}\alpha^{n-2}\,\beta + \frac{C_3}{n}\,\beta^{n-1}\right)}{\alpha}\end{array}\right]$$

The fundamental tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i\,\dot{\partial}_j\,L^2$ is given by

$$g_{ij} = P_0\,a_{ij} + P_0^*\,b_i\,b_j + P_{-1}^*\,(b_i\,y_j + b_j\,y_i) + q_{-2}^*\,y_i\,y_j \tag{2.6}$$

Where

$$P_0^* = q_0 + L_\beta^2 = L^{2-n}\left[\frac{C_3\,(n-1)\,(n-2)\,\alpha\,\beta^{n-3}}{n} + C_4(n-1)\,\beta^{n-2}\right] + (1-n)\,L_\beta^2 \quad + L_\beta^2$$

$$P_{-1}^* = q_{-1} + \frac{L_\alpha\,L_\beta}{\alpha} = \alpha^{-1}\left[L^{2-n}\left\{\frac{C_2\,(n-1)\,\alpha^{n-2}}{n} + \frac{C_3\,(n-1)\,\beta^{n-2}}{n}\right\} + (1-n)\,L_\alpha\,L_\beta\right] + \frac{L_\alpha\,L_\beta}{\alpha} \tag{2.7}$$

$$q_{-2}^* = P_{-2} + \left(\frac{L_\alpha}{\alpha}\right)^2 = \frac{L}{\alpha^2}\left[\begin{array}{c} L^{1-n}\left\{\left(C_1(n-1)\,\alpha^{n-2} + \frac{C_2\,(n-1)\,(n-2)\,\alpha^{n-3}\,\beta}{n}\right)\right\} + (1-n)L^{-1}L_\alpha^2 \\ -\frac{L^{1-n}\left(C_1\,\alpha^{n-1} + \frac{C_2\,(n-1)}{n}\alpha^{n-2}\,\beta + \frac{C_3}{n}\,\beta^{n-1}\right)}{\alpha}\end{array}\right] + \left(\frac{L_\alpha}{\alpha}\right)^2$$

Moreover, the reciprocal tensor g^{ij} of g_{ij} is given by [5]

$$g^{ij} = \frac{a^{ij}}{P_0} - S_0\,b^i\,b^j - S_1(b^i\,y^j + b^j\,y^i) - S_2\,y^i\,y^j \tag{2.8}$$

Where

$$b^i = a^{ij}b_j$$

$$S_0 = \frac{P_0\,P_0^* + \alpha^2\,[P_0^*\,q_{-2}^* - (P_{-1}^*)^2]}{P_0\left[P_0\,(P_0 + P_0^*\,b^2 + 2\,P_{-1}^*\,\beta + q_{-2}^*\,\alpha^2) + (\alpha^2\,b^2 - \beta^2)\left(-{P_{-1}^*}^2 + P_0^*\,q_{-2}^*\right)\right]}$$

$$J = P_0\left[P_0\,(P_0 + P_0^*\,b^2 + 2P_{-1}^*\,\beta + q_{-2}^*\,\alpha^2) + (\alpha^2 b^2 - \beta^2)\left(-{P_{-1}^*}^2 + P_0^*\,q_{-2}^*\right)\right]$$

$$S_0 = \frac{P_0\,P_0^* + \alpha^2[P_0^*\,q_{-2}^* - (P_{-1}^*)^2]}{J}$$

$$S_1 = \frac{P_{-1}^*\,P_0 + \beta\,[(P_{-1}^*)^2 - P_0^*\,q_{-2}^*]}{J} \tag{2.9}$$

$$S_2 = \frac{[P_0^*\,q_{-2}^* + b^2\,\{P_0^*\,q_{-2}^* - (P_{-1}^*)^2\}]}{J}$$

The h v-torsion tensor $c_{ijk} = \frac{1}{2}\,\dot{\partial}_k\,g_{ij}$ is given by [5]

$$2\,P_0\,c_{ijk} = P_{-1}^*\,(h_{ij}\,m_k + h_{jk}\,m_i + h_{ki}\,m_j) + r_1\,m_i\,m_j\,m_k \tag{2.10}$$

Where

$$r_1 = P_0 \frac{\partial P_0^*}{\partial \beta} - 3P_{-1}^* q_0 \ , \qquad m_i = b_i - \alpha^2 \beta y_i \qquad (2.11)$$

It is noted that the covariant vector m_i is a non-vanishing one, and is orthogonal to the element of support y^i.

Let $\{j \ {}^i \ k\}$ be the components of Christoffel's symbol of the associated Riemannian space R^h and ∇_k be covariant differentiation with respect to x^k relative to this Christoffel's symbol, we shall use the following tensors.

$$2E_{ij} = b_{ij} + b_{ji} \ , \qquad 2 F_{ij} = b_{ij} - b_{ji} \qquad (2.12)$$

Where $\quad b_{ij} = \nabla_j b_i$

If we donate the Cartan's connection C Γ as $\left(\Gamma_{jk}{}^{*i}, \Gamma_{0k}{}^{*i}, c_{jk}{}^{i}\right)$ then the difference tensor

$D_{jk}{}^{*i} = \Gamma_{jk}{}^{*i} - \{j \ {}^i \ k\}$ of $(\alpha, \beta) - metric$ space is given by [6]

$$D_{jk}{}^{i} = B^i E_{jk} + F_k^i B_j + F_j^i B_k + B_j^i b_{ok} + B_k^i b_{oj} - b_{0m} g^{im} B_{jk} - c_{jm}^i A_k^m - c_{km}^i A_j^m \qquad (2.13)$$

$$+ c_{jkm} A_s^m g^{is} + \left(c_{jm}^i c_{sk}^m + c_{km}^i c_{sj}^m - c_{jk}^m c_{ms}^i\right)$$

Where

$$B_k = P_0^* b_k + P_{-1}^* y_k \ , \qquad B^i = g^{ij} B_j \ , \qquad F_i^k = g^{kj} F_{ji}$$

$$B_{ij} = \left\{ \frac{P_{-1}^*}{P_0} \left(a_{ij} - \alpha^{-2} y_i y_j\right) + \frac{\partial P_0^*}{\partial \beta} m_i m_j \right\} / 2$$

$$B_i^k = g^{kj} B_{ji} \ , \qquad (2.14)$$

$$A_k^m = B_k^m E_{00} + B^m E_{k0} + B_k F_0^m + B_0 F_k^m \ ,$$

$$\lambda^m = B^m E_{00} + 2B_0 F_0^m \ , \qquad B_0 = B_i y^i \ ,$$

Here and in the following we denote 0 as contraction with y^i except for the quantities P_0^*, q_0 and c_0.

3. Induced Cartan Connection.

Let F^{n-1} be a hypersurface of F^n, given by the equation $x^i = x^i (u^\alpha)$ suppose that the matric of the projection factor $x_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}$ is of a rank $(n-1)$, the element of support y^i of F^n is to be taken tangential to F^{n-1} i.e.

$$y^i = x_\alpha^i (u) v^\alpha \ . \qquad (3.1)$$

Thus v^α is the element of support of F^{n-1} at the point u^α. The metric tensor $g_{\alpha\beta}$ and the hv-torsion tensor $C_{\alpha\beta\gamma}$ of F^{n-1} is defined by

$$g_{\alpha\beta} = g_{ij}\, X^i_\alpha\, X^j_\beta\,, \qquad C_{\alpha\beta\gamma} = C_{ijk}\, X^i_\alpha\, X^j_\beta\, X^k_\gamma \tag{3.2}$$

at each point u^α of F^{n-1} , a unit normal vector $N^i(u,\ v)$ is defined by

$$g_{ij} = \{x\,(u), y\,(u,v)\}\, X^i_\alpha\, N^j = 0\ , \qquad g_{ij}\,\{x\,(u),\ y\,(u,\ v)\}\, N^i\, N^j = 1 \tag{3.3}$$

As for the angular metric tensor h_{ij} we have

$$h_{\alpha\beta} = h_{ij}\, X^i_\alpha\, X^j_\beta\,, \qquad h_{ij}\, X^i_\alpha\, N^j = 0\,, \qquad h_{ij}\, N^i\, N^j = 1 \tag{3.4}$$

If $(X^\alpha_i\ ,\ N_i)$ denotes the inverse of $\left(B^i_\alpha\ ,\ N^i\right)$, then we have

$$X^\alpha_i = g^{\alpha\beta}\, g_{ij}\, B^j_\beta\,, \qquad X^i_\alpha\, X^\beta_i = \delta^\beta_\alpha\,, \qquad X^\alpha_i\, N^i = 0\,, \tag{3.5}$$

$$X^i_\alpha\, n_i = 0\,, \qquad N_i = g_{ij}\, N^j$$

$$X^i_\alpha\, \beta^\alpha_j + N^i\ N_j = \delta^i_j\,,$$

The induced connection I C $\Gamma = \left(\Gamma_{\beta\gamma}{}^{\alpha}\ ,\ G_\beta{}^{\alpha},\ C_{\beta\gamma}{}^{\alpha}\right)$ of F^{n-1} induced from the Cartan's connection

C $\Gamma = \left(\Gamma_{jk}{}^{*i}, \Gamma_{0k}{}^{*i}, C_{jk}{}^{i}\right)$ is given by [6]

$$\Gamma^{\ *\alpha}_{\ \beta\gamma} = X^\alpha_i\ \left(X^i_{\beta\gamma} + \Gamma_{jk}{}^{*i}\, X^j_\beta\ X^k_\gamma \right) + M^\alpha_\beta\, H_\gamma \tag{3.6}$$

$$G^{\ \alpha}_{\ \beta} = X^\alpha_i\ (\, X^i_{0\beta} + \Gamma_{0j}{}^{*i} + X^j_\beta\,) \tag{3.7}$$

$$C^\alpha_{\beta\gamma} = X^\alpha_i\ \ C_{jk}{}^{i}\, X^j_\beta\ X^k_\gamma \tag{3.8}$$

Where

$$M_{\beta\gamma} = N_i\, C^i_{jk}\, X^j_\beta\ X^k_\gamma\,, \qquad M^{\ \alpha}_{\ \beta} = g^{\alpha\gamma}\, M_{\beta\gamma} \tag{3.9}$$

$$H_\beta = N_i\ (\ X^i_{0\beta} + \Gamma_{0j}{}^{*i}\, X^j_\beta\ ) \tag{3.10}$$

and $\quad X^i_{\beta\gamma} = \frac{\partial X^i_\beta}{\partial u^\gamma}\,, \qquad X^i_{0\beta} = X^i_{\alpha\beta}\, v^\alpha\,,$

the quantities $M_{\beta\gamma}$ and H_β are called second fundamental v-tensor and normal curvature vector respectively [6].

The second fundamental v-tensor $H_{\beta\gamma}$ is defined as [6]

$$H_{\beta\gamma} = N_i\ (\ X^{\ i}_{\ \beta\gamma} + \Gamma_{jk}{}^{*i}\, X^{\ j}_{\ \beta}\ X^k_\gamma) + M_\beta\, H_\gamma \tag{3.11}$$

Where

$$M_\beta = N_i\, C^i_{jk}\, X^j_\beta\ N^k \tag{3.12}$$

The relative h- and v-covariant derivatives of projection factor X^i_α with respect to I C Γ are given by

$$X^i_{\alpha|\beta} = H_{\alpha\beta}\ \ N^i\,, \qquad X^i_{\alpha/\beta} = M_{\alpha\beta}\ N^i \tag{3.13}$$

The equation (3.11) shows that h is generally not symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta\, H_\gamma - M_\gamma\, H_\beta\ . \tag{3.14}$$

Furthermore (3.10), (3.11) and (3.12) yield

$$H_{0\gamma} = H_\gamma , \qquad H_{\gamma 0} = H_\gamma + M_\gamma H_0 , \tag{3.15}$$

We quote the following Lemma which is due to Matsumoto [6]

Lemma (3.1): The normal curvature $h_0 = H_\beta v^\beta$ vanishes if and only if the normal curvature vector H_β vanishes.

The hyperplanes of first, second and third kinds are defined [6] we only quote the following.

Lemma (3.2): A hypersurface F^{n-1} is a hyperplane of the first kind if and only if $H_\alpha = 0$.

Lemma (3.3): A hypersurface F^{n-1} is a hyperplane of the third kind with respect to the connection C Γ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.

Lemma (3.4): A hypersurface F^{n-1} is a hyperplane of the third kind with respect to the connection C Γ if and only if $H_\alpha = 0$, $M_{\alpha\beta} = H_{\alpha\beta} = 0$.

4. The hypersurface F^{n-1} (c)

Let us consider a special (α, β)-metric (1.1) with a gradient $b_i(x) = \partial_i b$ for a scalar function $b(x)$ and consider a hypersurface $F^{n-1}(c)$ which is given by the equation $b(x) = c$(constant).

From the parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get $\frac{\partial b(x(u))}{\partial u^\alpha} = 0 = b_i x_\alpha^i$, so that $b_i(x)$ are regarded as covariant components of a normal vector field of $F^{n-1}(c)$.

Therefore along the $F^{n-1}(c)$ we have

$$b_i x_\alpha^i = 0 , \qquad b_i y^i = 0 \tag{4.1}$$

In general the induced metric $\underline{L}(u, v)$ from the metric (1.1) is given by

$$\underline{L}^n(u, v) = c_1 \left\{ a_{ij}(x(u)) X_\alpha^i X_\beta^j v^\alpha v^\beta \right\}^{n/2}$$

therefore the induced metric of $F^{n-1}(c)$ becomes

$$\underline{L}(u, v) = \sqrt{c_1 a_{\alpha\beta}(u) v^\alpha v^\beta} , \qquad a_{\alpha\beta} = a_{ij}(x(u)) X_\alpha^i X_\beta^i \tag{4.2}$$

Which is a Riemannian metric, at the point of $F^{n-1}(c)$ from (2.5), (2.7) and (2.9)

we have

$$P_0 = C_1^{2/n} , \qquad q_0 = \frac{(1-n)\, C_1^{2(1-n)/n}\, C_2^2}{n^2} , \qquad q_{-1} = 0 ,$$

$$P_{-2} = -C_1^{2/n}\, \alpha^{-2}, \qquad P_0^* = \frac{(2-n)\, C_1^{2(1-n)/n}\, C_2^2}{n^2} , \qquad P_{-1}^* = \frac{C_1^{(2-n)/n)}\, C_2\, \alpha^{-1}}{n}$$

$$q_{-2}^* = 0 ,$$

$$\mathrm{J} = \frac{C_1^{\left(\frac{6}{n}-2\right)}}{n^2} [C_1^2\, n^2 + (1-n)\, C_2^2\, b^2]$$

$$S_0 = \frac{(1-n)\, C_2^2}{C_1^{2/n}\,[C_1^2\, n^2 + (1-n)\, C_2^2\, b^2]} \tag{4.3}$$

$$S_1 = \frac{C_1\ C_2\, n}{\alpha\, C_1^{2/n}\,[C_1^2\, n^2 + (1-n)\, C_2^2\, b^2]}$$

$$S_2 = -\frac{b^2\ C_2^2}{\alpha^2\ C_1^{2/n}\,[C_1^2\, n^2 + (1-n)\, C_2^2\, b^2]}$$

Therefore from (2.8) we get

$$g^{ij} = \frac{a^{ij}}{C_1^{2/n}} - \frac{C_2^2\,(1-n)}{C_1^{2/n}\,[C_1^2\, n^2 + (1-n) C_2^2 b^2]} b^i\, b^j - \frac{C_1\, C_2\, n}{\alpha\, C_1^{2/n}\ [C_1^2\, n^2 + (1-n)\, C_2^2\, b^2]} \left(b^i y^j + b^j y^i\right) \tag{4.4}$$

$$+ \frac{b^2\, C_2^2}{\alpha^2\, C_1^{2/n}\,[C_1^2\, n^2 + (1-n)\, C_2^2\, b^2]}\ y^i\, y^j$$

Thus along F^{n-1} (4. 1) and (4.4) lead to

$$g^{ij} b_i b_j = \frac{b^2\, C_1^2\, n^2}{C_1^{2/n}\ [C_1^2\, n^2 + (1-n)\, C_2^2 b^2]} \qquad \text{therefore we get}$$

$$b_i\,(x\,(u)) = \sqrt{\frac{b^2\, C_1^2\, n^2}{C_1^{2/n}\ [C_1^2\, n^2 + (1-n)\, C_2^2 b^2]}}\, N_i\,, \qquad b^2 = a^{ij} b_i\, b_j \tag{4.5}$$

Again from (4.4) and (4.5) we get

$$b^i = a^{ij} b_j = \sqrt{\frac{b^2 C_1^{2/n}\,[C_1^2\, n^2 + (1-n)\, C_2^2\, b^2]}{C_1^2\, n^2}}\quad N^i + \frac{C_2\, b^2}{\alpha\, n\, C_1}\ y^i \tag{4.6}$$

Hence we have the following,

Theorem 4.2 : Let F^n be a Finsler space with (α, β)- metric (1.1) and $b_i(x) = \partial_i b\,(x)$. Let $F^{n-1}(c)$ be a hypersurface of F^n given by $b\,(x) = c$ (constant) suppose the Riemannian metric $a_{ij}(x)\, \delta\, x^i\, \delta\, x^j$ be positive definite and b_i be non-zero field then the induced metric an $F^{n-1}(c)$ is a Riemannian metric given by (4.2) and relation (4.5) and (4.6) hold.

Along $F^{n-1}(c)$, the angular metric tensor and metric tensor are given by

$$h_{ij} = C_1^{2/n}\, a_{ij} - C_1^{\frac{2}{n}}\, \alpha^{-2}\, y_i\, y_j + \frac{(1-n)\, C_1^{\frac{2(1-n)}{n}}\, C_2^2}{n^2}\ b_i\, b_j \tag{4.7}$$

$$g_{ij} = C_1^{2/n}\, a_{ij} + \frac{(2-n)\, C_1^{\frac{2(1-n)}{n}}\, C_2^2}{n^2}\ b_i\, b_j + \frac{C_1^{\frac{2-n}{n}}\, C_2\, \alpha^{-1}}{n} \left(b_i\, y_j + b_j\, y_i\right) \tag{4.8}$$

From (4.1), (4.7) and (3.4) it follows that if $h_{\alpha\beta}^{(a)}$ denote the angular metric tensor of Riemannian metric $a_{ij}(x)$ then along $F^{n-1}(c)$, $h_{\alpha\beta} = C_1{}^{2/n}\ h_{\alpha\beta}^{(a)}$. From (2.7) we get $\frac{\partial P_0^*}{\partial \beta} = 0$ thus along $F^{n-1}(c)$, (2.11) and (4.3) give

$$r_1 = \frac{C_1^{(4-3n)/n}\, \alpha^{-1}\, C_2^3}{n^3}\ (n-2)(2n-1), \qquad m_i = b_i$$

Therefore hv-torsion tensor becomes

$$C_{ijk} = \frac{C_2}{2n\,\alpha\,C_1}\left[\left(b_i\,h_{jk} + b_j\,h_{ki} + b_k\,h_{ij}\right) + \frac{C_1^{(2-3n)/n}\,C_2^2}{n^2}\,(n-1)\,(2n-1)\,b_i\,b_j\,b_k\right] \quad (4.9)$$

Therefore (3.4), (3.9), (3.12), (4.1), (4.5) and (4.9) gives

$$M_{\alpha\beta} = \frac{C_2}{2n\,\alpha\,C_1}\sqrt{\frac{b^2\,C_1^2\,n^2}{C_1^{\frac{2}{n}}\,\left[C_1^2\,n^2 + (1-n)\,C_2^2 b^2\right]}}\;\;h_{\alpha\beta}\;\;, M_\alpha = 0 \quad (4.10)$$

Hence from (3.14) it follows that $H_{\alpha\beta}$ is symmetric.

Theorem 4.2: The second fundamental tensor v-tensor of $F^{n-1}(c)$, is given by (4.10) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.

Next from (4.1) we get $b_{i|\beta}\,X^{\,i}_{\,\alpha} + b_i\,X^{\,i}_{\,\alpha|\beta} = 0$ therefore from (3.13) and the fact that

$b_{i|\beta} = b_{i|j}\,X_\beta^j + b_{i|j}\,N^j\,H_\beta$ [6] we get

$$b_{i|j}\,X_\alpha^i\,X_\beta^j + b_{i|j}\,X_\alpha^i\,N^j\,H_\beta + H_{\alpha\beta}\,b_i\,N^i = 0 \quad (4.11)$$

Since $b_{i|j} = -b_h\,C_{ij}^h$ from (3.12) , (4.5) and (4.10) we get

$$b_{i|j}\,X_\alpha^i\;\;N^j = -\sqrt{\frac{b^2\,C_1^2\,n^2}{C_1^{\frac{2}{n}}\,\left[C_1^2\,n^2 + (1-n)\,C_2^2 b^2\right]}}\;\;M_\alpha = 0$$

Thus (4.11) gives

$$\sqrt{\frac{b^2\,C_1^2\,n^2}{C_1^{\frac{2}{n}}\,\left[C_1^2\,n^2 + (1-n)\,C_2^2 b^2\right]}}\;\;H_{\alpha\beta} + b_{i|j}\,X_\alpha^i\,X_\beta^j = 0 \quad (4.12)$$

It is noted that $b_{i|j}$ is symmetric. Furthermore contracting (4.12) with v^β and v^α respectively and using (3.1), (3.15) we get,

$$\sqrt{\frac{b^2\,C_1^2\,n^2}{C_1^{\frac{2}{n}}\,\left[C_1^2\,n^2 + (1-n)\,C_2^2 b^2\right]}}\;\;H_\alpha + b_{i|j}\,X_\alpha^i\,y^j = 0$$

$$\sqrt{\frac{b^2\,C_1^2\,n^2}{C_1^{\frac{2}{n}}\,\left[C_1^2\,n^2 + (1-n)\,C_2^2 b^2\right]}}\;\;H_0 + b_{i|j}\;y^i\,y^j = 0 \quad (4.13)$$

In view of Lemmas (3.1) and (3.2), the hypersurface $F^{n-1}(c)$ is a hyperplane of the first kind if only if $H_0 = 0$.

Thus from (4.13) it follows that $F^{n-1}(c)$ is the hyperplane of first kind if and only if

$b_{i|j}\;\;y^i\,y^j = 0$. This b_{ij} being covariant derivative with respect to Cartan's connection C Γ of F^n, it may depend on y^i. On the other hand $\nabla_j\,b_i = b_j$ is the covariant derivative with respect to the Riemannian connection $\{_j{}^i{}_k\}$ constructed from $a_{ij}(x)$, therefore b_{ij} does not depend on y^i. We shall

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consider the difference $b_{i|j} - b_{ij}$ in the following. The difference tensor $D^{\ i}_{jk} = \Gamma^{\ i}_{jk} - \{j\ {}^{i}\ k\}$ is given by (2.13), since b_i is a gradient vector, from (2.12) we have

$$E_{ij} = b_{ij}\,, \qquad F_{ij} = 0, \qquad F^{\ i}_{j} = 0$$

Thus (2.13) reduces to

$$D^{\ i}_{jk} = B^i\, b_{jk} + B^{\ i}_{j}\, b_{0k} + B^{\ i}_{k}\, b_{0j} - b_{0m}\, g^{im}\, B_{jk} - C^{\ i}_{jm}\, A^{\ m}_{k} - C^{\ i}_{km}\, A^{\ m}_{j} + C_{jkm}\, A^{\ m}_{s} g^{is}$$
$$+\lambda^s\left[C^{\ i}_{jm}\, C^{\ m}_{sk} + C^{\ i}_{km}\, C^{\ m}_{sj} - C^{\ m}_{jk}\, C^{\ i}_{ms}\right] \tag{4.14}$$

But in view of (4.3) and (4.4) the expression (2.14) reduce to

$$B_i = \frac{C_1^{2(1-n)/n}\, C_2^2\,(2-n)}{n^2}\, b_i + \left(\frac{C_1^{(2-n)/n}\, C_2\,\alpha^{-1}}{n}\right) y_i\,,$$

$$B^i = \frac{(1-n)\, C_2^2}{\left[C_1^2\, n^2 + (1-n)\, C_2^2\, b^2\right]}\, b^i + \frac{C_1\, C_2\, n}{\alpha\{C_1^2\, n^2 + (1-n)\, C_2^2\, b^2\}}\, y^i$$

$$B_{ij} = \frac{C_2}{2\alpha n C_1}\left(a_{ij} - \alpha^{-2}\, y_i\, y_j\right)$$

$$B^{\ i}_{j} = \frac{C_2}{2n\alpha C_1}\left(\delta^i_j - \alpha^{-2}\, y^i\, y_j\right) - \frac{C_2^3(1-n)}{2n\alpha C_1\{C_1^2\, n^2 + (1-n)\, C_2^2\, b^2\}}\, b^i\, b_j - \frac{C_2^2}{2n\alpha^2\{C_1^2\, n^2 + (1-n)\, C_2^2\, b^2\}}\, b_j\, y^i$$

(4.15)

$$A^{\ m}_{k} = B^{\ m}_{k} b_{00} + B^m b_{k0}\,, \qquad \lambda^m = B^m b_{00}$$

By the virtue of (4.1), we have $B^{\ i}_{0} = 0,$ $B_{i0} = 0$ which give $A^{\ m}_{0} = B^m b_{00}$, therefore we have

$$D^{\ i}_{j0} = B^i\, b_{j0} + B^{\ i}_{j}\, b_{00} - B^m\, C^{\ i}_{jm}\, b_{00} \tag{4.16}$$

$$D^{\ i}_{00} = B^i\, b_{00} = \left[\frac{(1-n)\, C_2^2}{\{C_1^2\, n^2 + (1-n)\, C_2^2\, b^2\}}\, b^i + \frac{C_1\, C_2\, n}{\alpha\{C_1^2\, n^2 + (1-n)\, C_2^2\, b^2\}}\, y^i\right] b_{00} \tag{4.17}$$

Thus paying attention to (4.1) along $F^{n-1}(c)$, we finally get

$$b_i\, D^i_{j0} = \frac{(1-n)\, C_2^2\, b^2}{\{C_1^2\, n^2 + (1-n)\, C_2^2\, b^2\}}\, b_{j0} + \frac{C_1\, C_2\, n}{2\alpha\{C_1^2\, n^2 + (1-n)\, C_2^2\, b^2\}}\, b_j\, b_{00} - \frac{(1-n)\, C_2^2\, b^m}{\{C_1^2\, n^2 + (1-n)\, C_2^2\, b^2\}}\, C^{\ i}_{jm}\, b_i\, b_{00} \tag{4.18}$$

$$b_i\, D^i_{00} = \frac{(1-n)\, C_2^2\, b^2}{\{C_1^2\, n^2 + (1-n)\, C_2^2\, b^2\}}\, b_{00} \tag{4.19}$$

From (3.12), (4.5), (4.6) and (4.10) it follows that

$$b^m\, b_i\, C^i_{jm}\, X^j_\alpha = b^2\, M_\alpha = 0$$

Therefore the relation

$b_{i/j} = b_{ij} - b_r D^r_{ij}$ and equation (4.1), (4.18), (4.19) give

$$b_{i/j} X^i_\alpha y^j = b_{i0} X^i_\alpha - b_r D^r_{i0} X^i_\alpha = \frac{C_1^2 n^2}{[C_1^2 n^2 + (1-n) C_2^2 b^2]} b_{i0} X^i_\alpha$$

$$b_{i/j} y^i y^j = b_{00} - b_r D^r_{00} = \frac{C_1^2 n^2}{[C_1^2 n^2 + (1-n) C_2^2 b^2]} b_{00}$$

Consequently (4.13) may be written as

$$\sqrt{b^2} H_\alpha + c_1 n \sqrt{\frac{C_1^{2/n}}{[C_1^2 n^2 + (1-n) C_2^2 b^2]}} b_{i0} X^i_\alpha = 0$$

$$\sqrt{b^2} H_0 + c_1 n \sqrt{\frac{C_1^{2/n}}{[C_1^2 n^2 + (1-n) C_2^2 b^2]}} b_{00} = 0 \qquad (4.20)$$

Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$, where b_{ij} does not depend on y^i . Since y^i is to satisfy (4.1) the condition is written as

$b_{ij} y^i y^j = (b_i y^i)(d_j y^j)$ for some $d_j(x)$ so that we have

$$2 b_{ij} = b_i d_j + b_j d_i \qquad (4.21)$$

From (4.1) and (4.2) it follows that

$$b_{00} = 0, \qquad b_{ij} X^i_\alpha X^j_\beta = 0, \qquad b_{ij} X^i y^j = 0.$$

Hence (4.20) gives $H_\alpha = 0$. Again from (4.1), (4.21) and (4.15) we get

$$b_{i0} b^i = \frac{d_0 b^2}{2}, \qquad \lambda^m = 0, \qquad A^i_j X^j_\beta = 0, \qquad \text{and } B_{ij} X^i_\alpha X^j_\beta = \frac{C_2}{2n\alpha C_1} h_{\alpha\beta}.$$

Thus (3.9), (4.4), (4.5), (4.6), (4.10) and (4.4) give

$$b_r D^r_{ij} X^i_\alpha X^j_\beta = -\frac{n^3 C_1^3 C_2 b^2 d_0}{4\alpha C_1^{2/n} \{C_1^2 n^2 + (1-n) C_2^2 b^2\}^2} h_{\alpha\beta}$$

Therefore the equation (4.12) reduces to

$$\sqrt{\frac{b^2 C_1^2 n^2}{C_1^{2/n} K}} H_{\alpha\beta} + \frac{n^3 C_1^3 C_2 b^2 d_0}{4\alpha C_1^{2/n} K^2} h_{\alpha\beta} = 0 \qquad (4.22)$$

where

$$K = \{C_1^2 n^2 + (1-n) C_2^2 b^2\}$$

Hence the hypersurface $F^{n-1}(c)$ is umbilic.

Theorem 4.3 : The necessary and sufficient condition for $F^{n-1}(c)$ to be hyperplane of the first kind is (4.21) and in this case the second fundamental tensor of $F^{n-1}(c)$ is proportional to its angular metric tensor.

In view of Lemma (3.3) $F^{n-1}(c)$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$ thus from (4.22) we $d_0 = d_i(x) y^i = 0$, therefore there exist a function $E(x)$ such that

$d_i(x) = E(x)\, b_i(x)$ thus (4.21) gives

$$b_{ij} = E\, b_i\, b_j \;. \tag{4.23}$$

Theorem 4.4 : The necessary and sufficient condition for $F^{n-1}(c)$ to be a hyperplane of the second kind is (4.23).

Finally (4.10) and Lemma (3.4) show that $F^{n-1}(c)$ does not become a hyperplane of the third kind.

Theorem 4.5 : The hypersurface $F^{n-1}(c)$ is not a hyperplane of the third kind

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